CSE 250
Data Structures

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Day 26
AVL Trees

Announcements

- WA2 due tonight @ 11:59pm

BST Operations

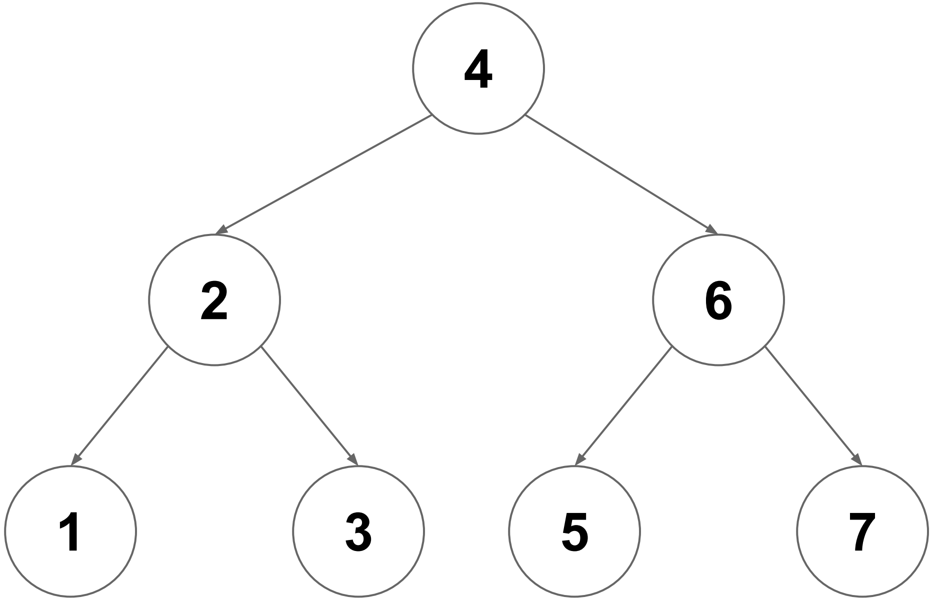
Operation	Runtime
`find`	$O(d)$
`insert`	$O(d)$
`remove`	$O(d)$

*What is the runtime in terms of **n**?* $\boldsymbol{O(n)}$

$$\log(n) \le d \le n$$

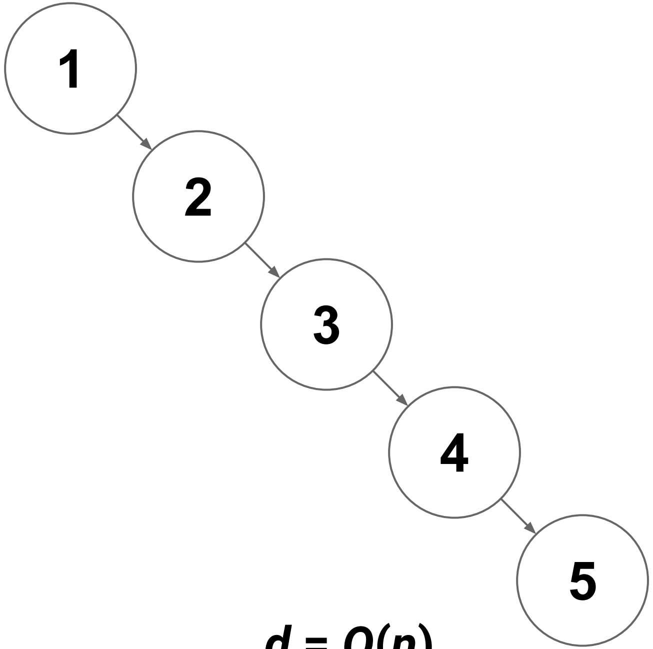
Tree Depth vs Size

If height(left) ≈ height(right)



$d = O(\log(n))$

If height(left) ≪ height(right)



$d = O(n)$

Balanced Trees

Balanced Trees are good: Faster **`find`**, **`insert`**, **`remove`**

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What do we mean by balanced?

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What do we mean by balanced? **|height(left) - height(right)| ≤ 1**

Balanced Trees

Balanced Trees are good: Faster **`find`**, **`insert`**, **`remove`**

What do we mean by balanced? **|height(left) - height(right)| ≤ 1**

How do we keep a tree balanced?

Balanced Trees - Two Approaches

Option 1

Keep left/right subtrees within **+/-1** of each other in height

(add a field to track amount of "imbalance")

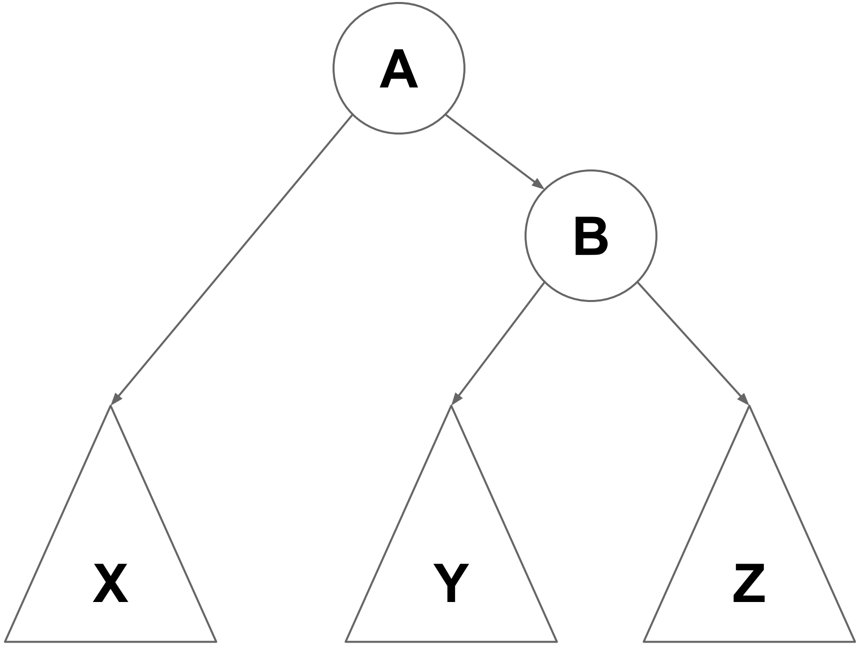
Option 2

Keep leaves at some minimum depth (***d*/2**)

(Add a color to each node marking it as "red" or "black")

Ok…but how do we enforce this…?

Rebalancing Trees (rotations)



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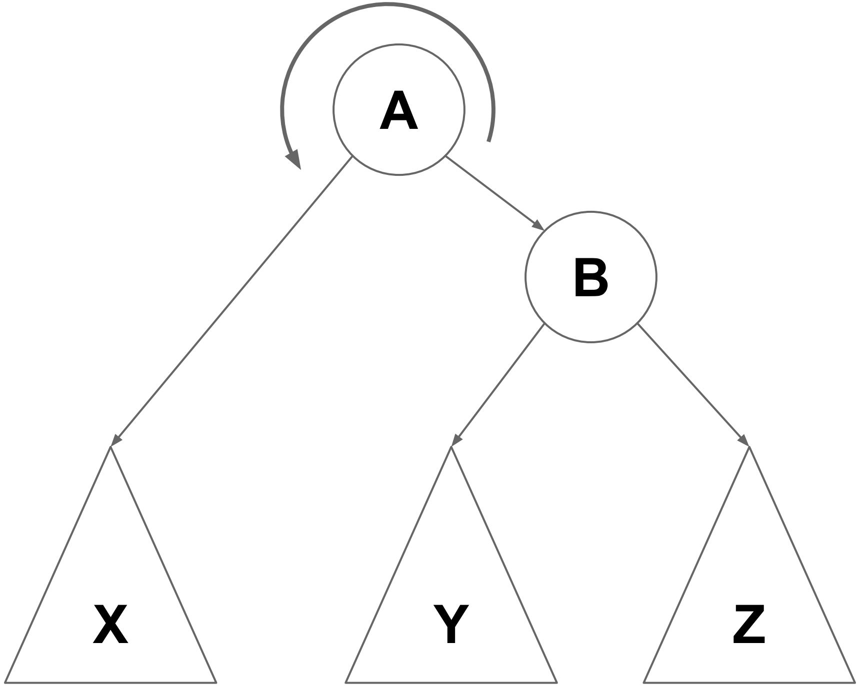
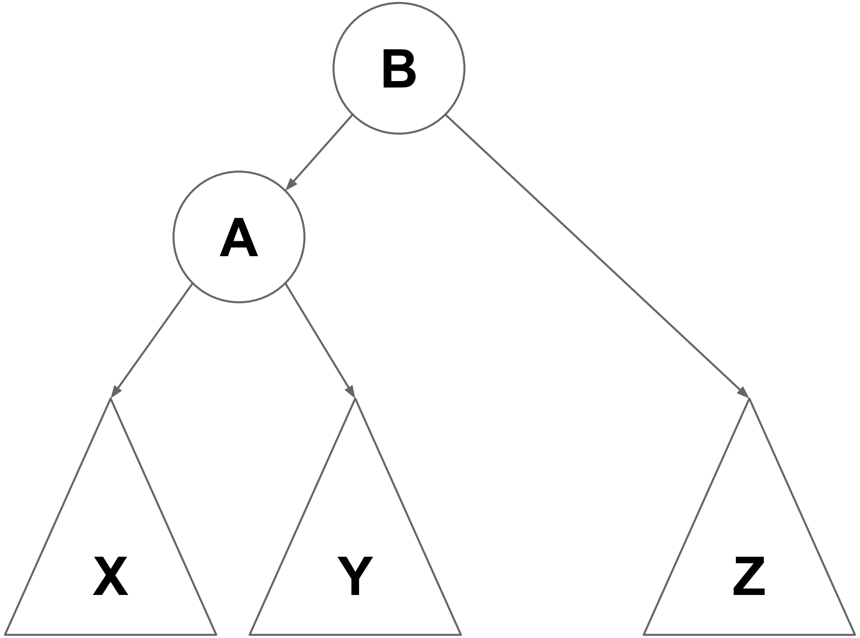


Rotate(A, B)

Rebalancing Trees (rotations)

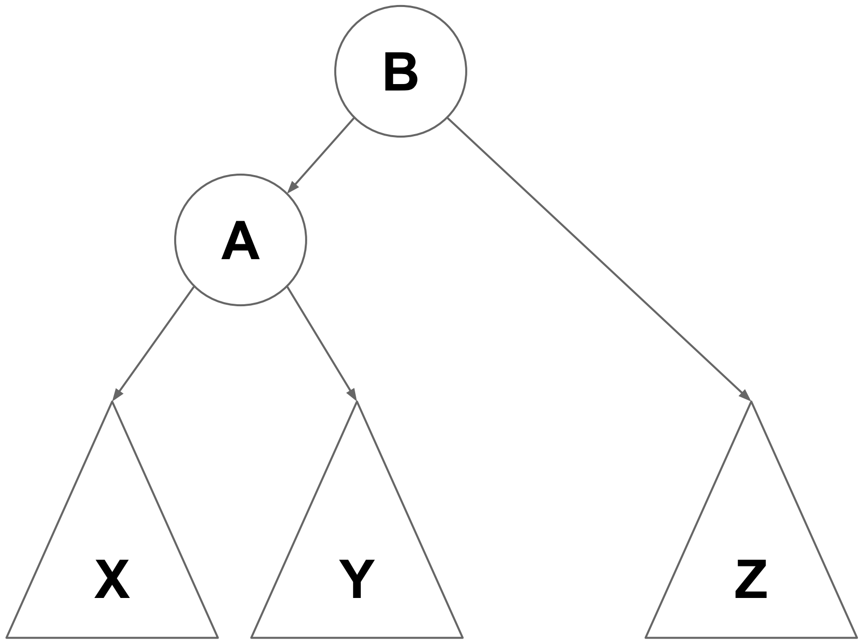


Rotate(A, B)

Rebalancing Trees (rotations)

A became **B**'s left child

B's left child became **A**'s right child



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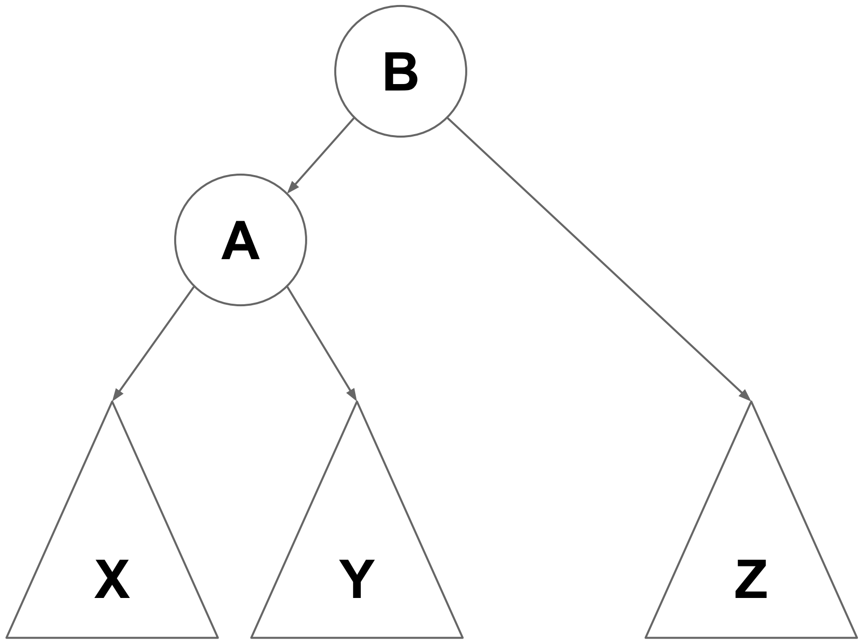
Is ordering maintained?



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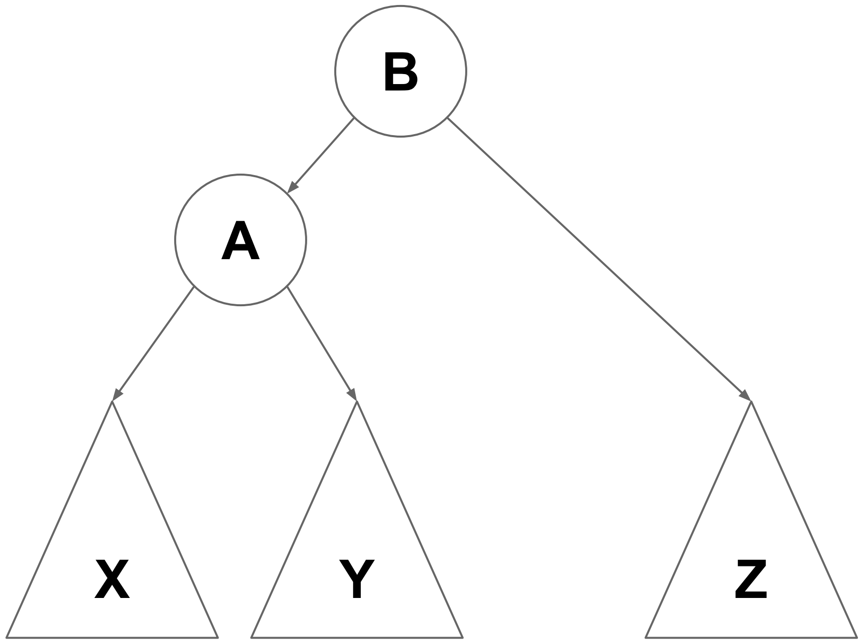

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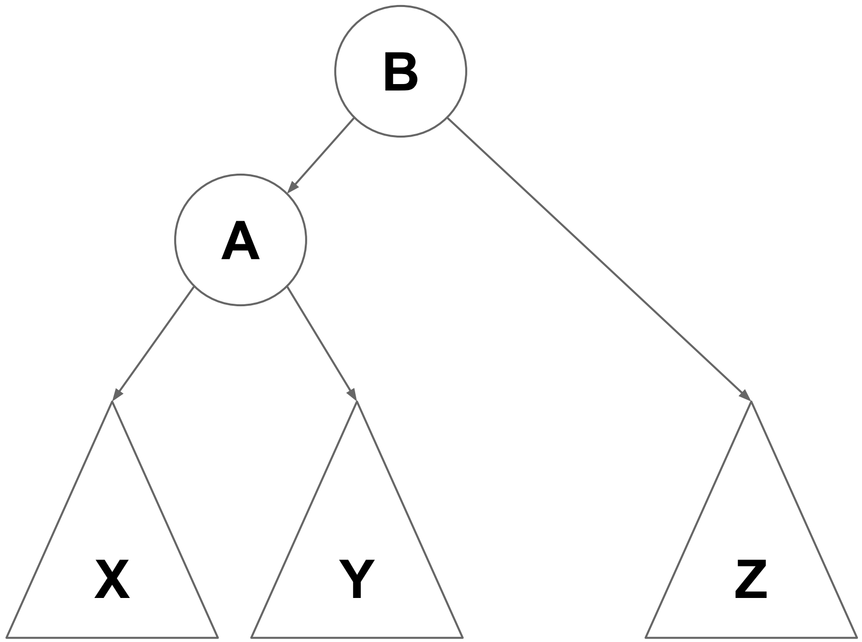
Complexity?



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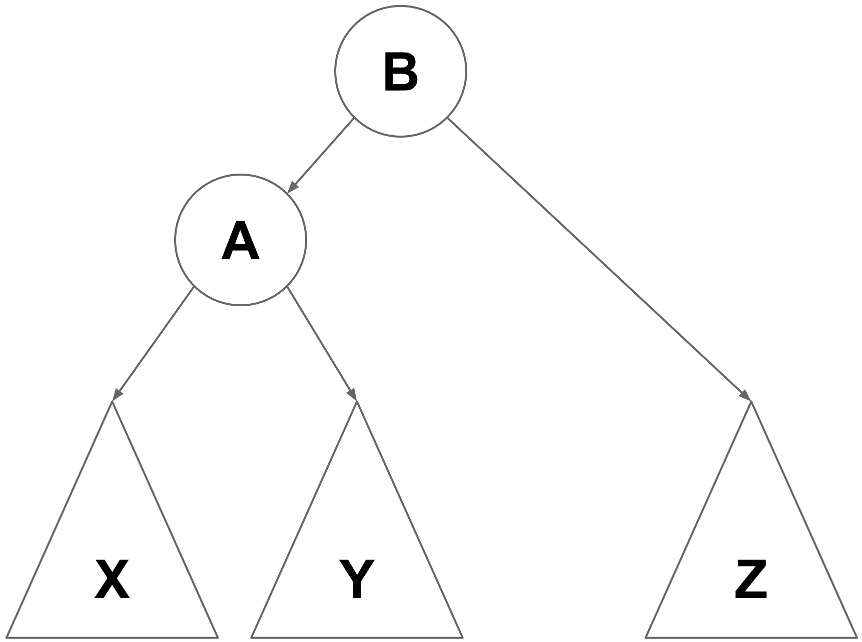
Complexity? ***O*(1)**



Rotate(A, B)

Rebalancing Trees (rotations)

A became **B**'s left child

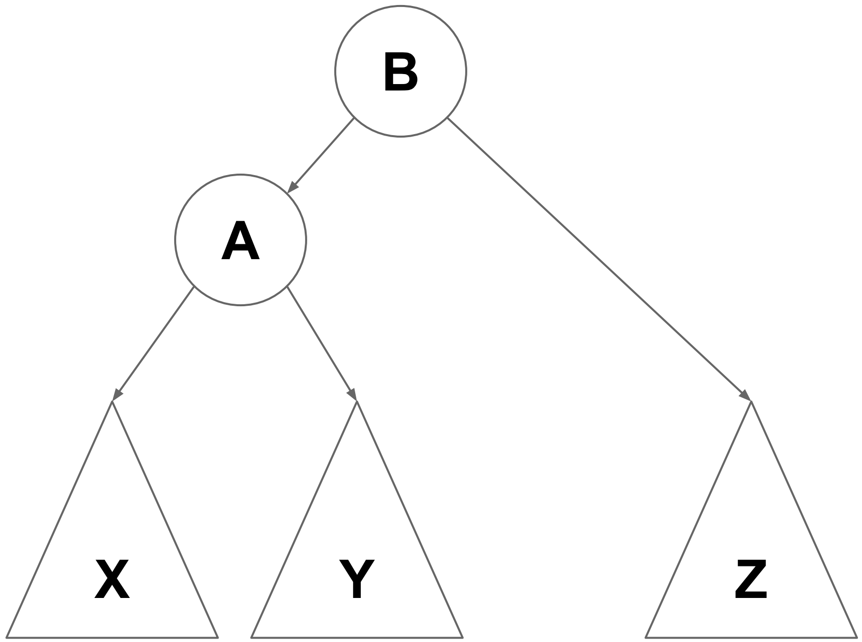
B's left child became **A**'s right child

Is ordering maintained? **Yes!**

Complexity? ***O*(1)**

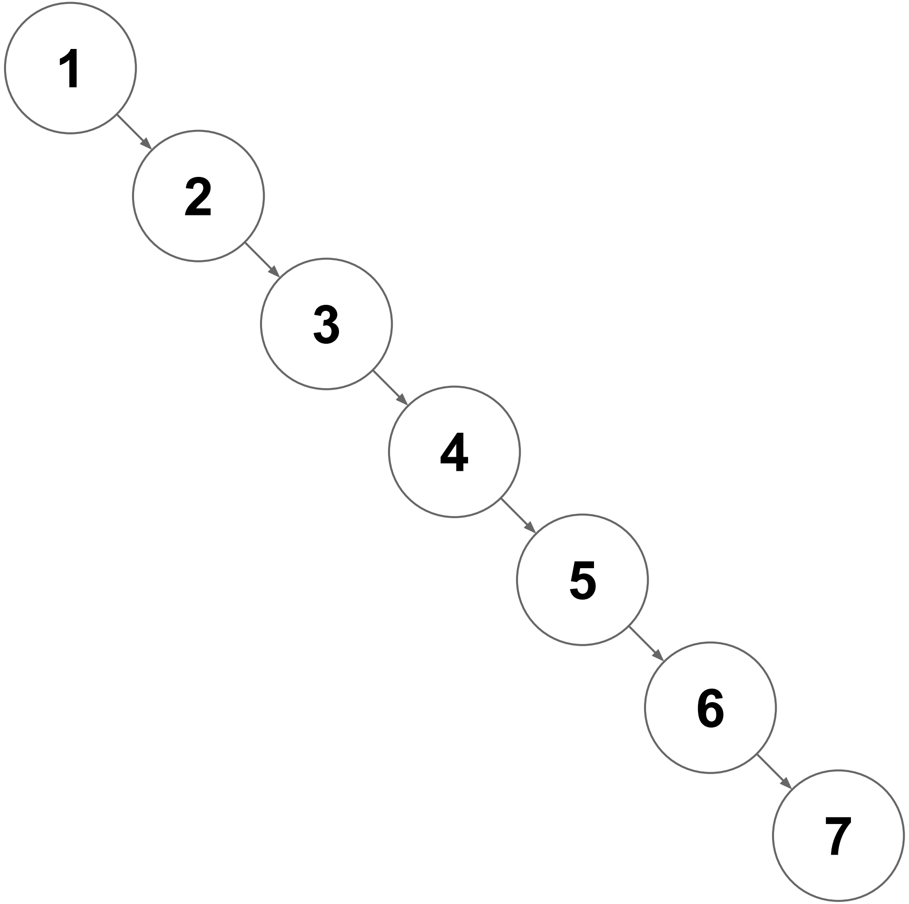
This is called a left rotation

(right rotation is the opposite)



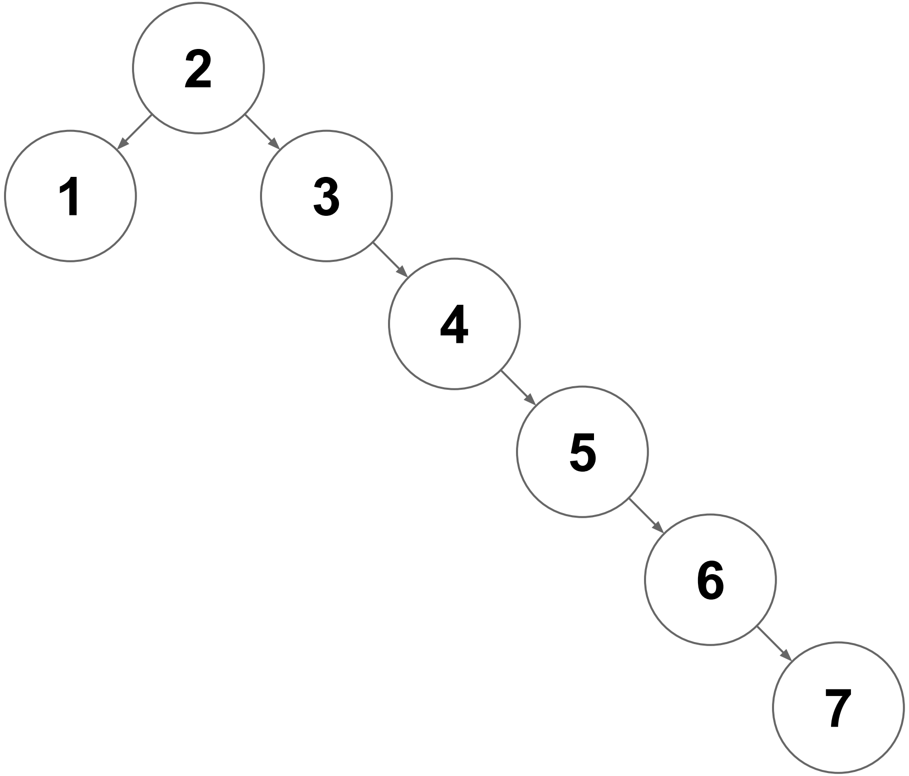
Rotate(A, B)

Rebalancing Trees



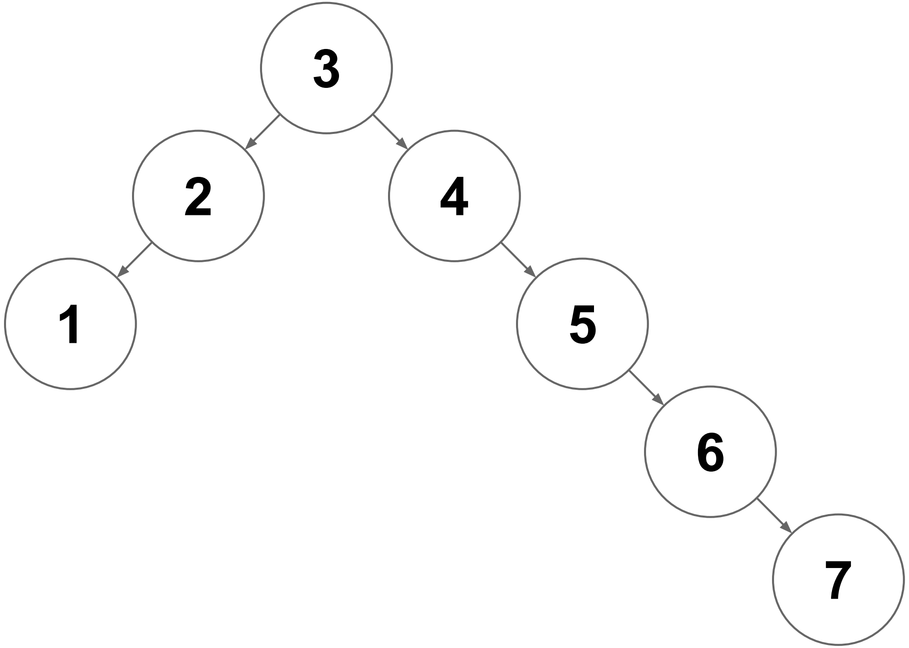
Rebalancing Trees

`Rotate(1,2)`



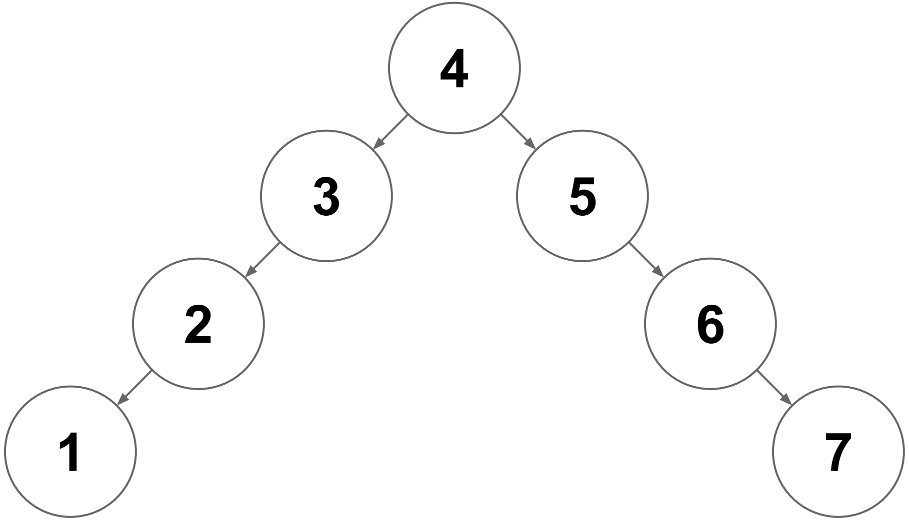
Rebalancing Trees

`Rotate(2,3)`



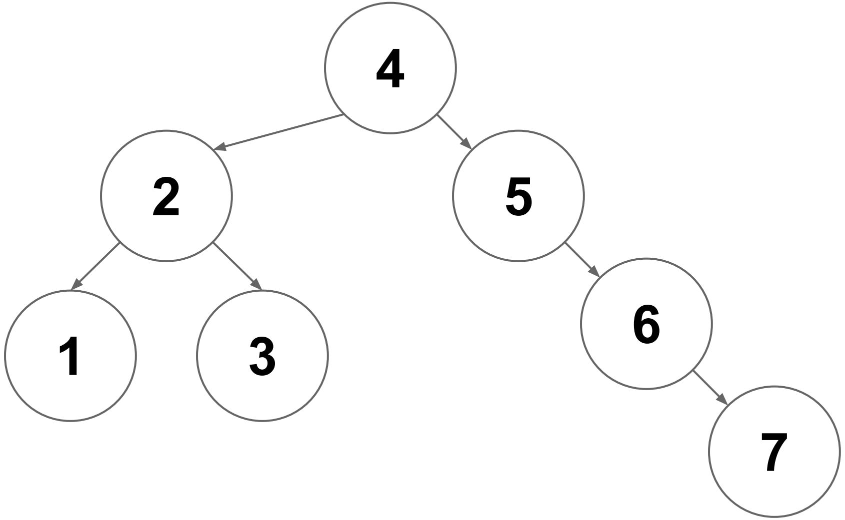
Rebalancing Trees

`Rotate(3,4)`



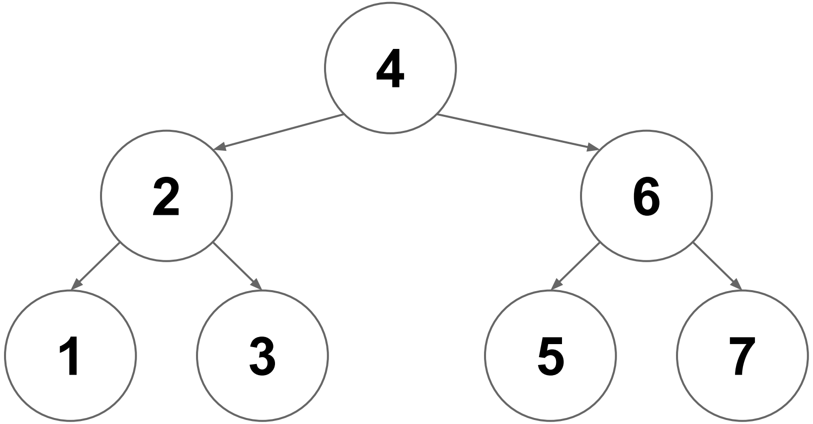
Rebalancing Trees

`Rotate(3,2)`



Rebalancing Trees

`Rotate(5,6)`



AVL Trees

AVL Trees

An **AVL tree** (**A**delson-**V**elsky and **L**andis) is a ***BST*** where every subtree is depth-balanced

Remember: Tree depth = height(root)

Balanced: |height(root.left) - height(root.right)| ≤ 1

AVL Trees

Define **balance(*v*) = height(*v.right*) - height(*v.left*)**

Goal: Maintaining **balance(*v*) ∈ { -1, 0, 1 }**

- **balance(*v*) = 0** → "***v*** is balanced"
- **balance(*v*) = -1** → "***v*** is left-heavy"
- **balance(*v*) = 1** → "***v*** is right-heavy"

AVL Trees

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What does enforcing this gain us?

AVL Trees - Depth Bounds

Question: Does the AVL property result in any guarantees about depth?

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YES! Depth balance forces a maximum possible depth of **log(*n*)**

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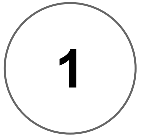
YES! Depth balance forces a maximum possible depth of **log(*n*)**

Proof Idea: An AVL tree with depth ***d*** has "enough" nodes

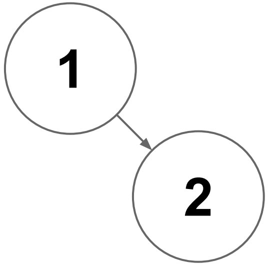
AVL Trees - Depth Bounds

Let **minNodes(*d*)** be the minimum number of nodes an in AVL tree of depth ***d***

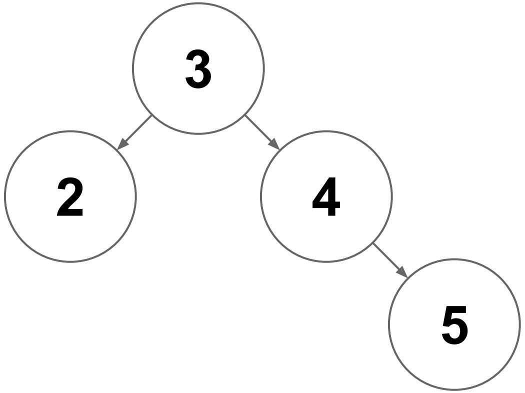
minNodes(0) = 1



minNodes(1) = 2

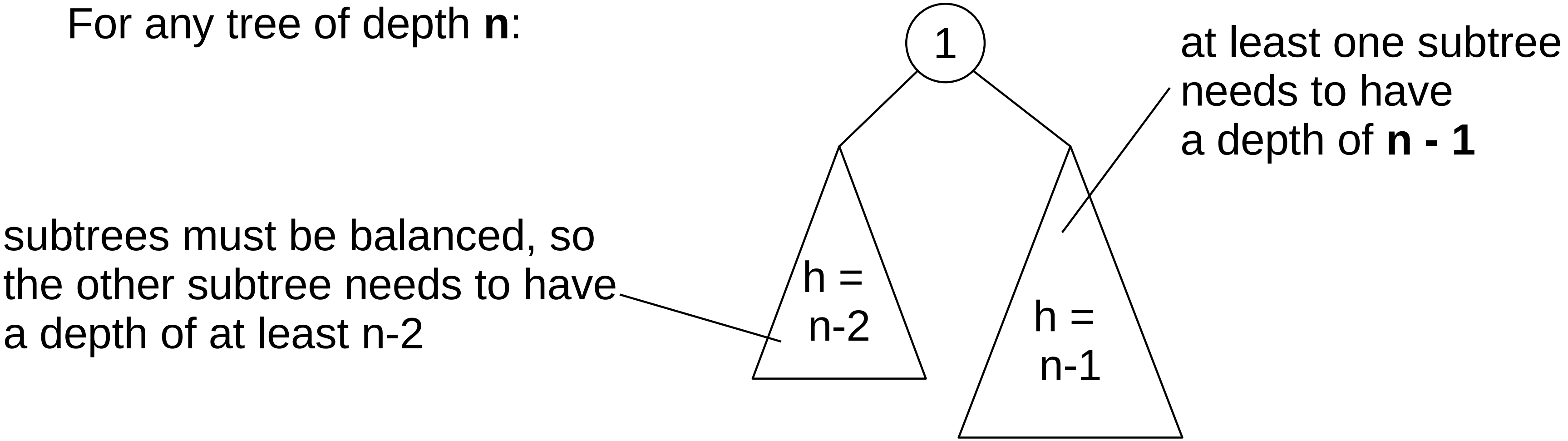


minNodes(2) = 4



AVL Trees

For any tree of depth **n**:



$$\text{minNodes}(n) = \begin{cases} \end{cases}$$

Enough Nodes?

- For $d > 1$
  - $\text{minNodes}(d) = 1 + \text{minNodes}(d-1) + \text{minNodes}(d-2)$
  - This is the Fibbonacci Sequence!
    - $\text{minNodes}(d) = \text{Fib}(d+3)-1$
    - $\text{Fib}(0), \text{Fib}(1), \text{Fib}(2), \ldots = 0, 1, 1, 2, 3, 5, 8, \ldots$
  - $\text{minNodes}(d) = \Omega(1.5^d)$

Enough Nodes?

- minNodes(d) = $\Omega(1.5^d)$

$$n \geq c1.5^d$$

$$\frac{n}{c} \geq 1.5^d$$

$$\log_2\left(\frac{n}{c}\right) \geq \log_2\left(1.5^d\right)$$

$$\log_2\left(\frac{n}{c}\right) \geq \log_{1.5}(1.5^d)\log_2 1.5$$

$$\log_2\left(\frac{n}{c}\right) \geq d\log_2(1.5)$$

$$\frac{\log_2\left(\frac{n}{c}\right)}{\log_2(1.5)} \geq d$$

constant

$$\frac{\log_2(n)}{\log_2(1.5)} - \frac{\log_2(c)}{\log_2(1.5)} \geq d$$

$$O\left(\log_2(n)\right) \geq d$$

A tree with n nodes and the AVL constraint has logarithmic depth in n

Enforcing the AVL Constraint

- Computing balance() on the fly is expensive
  - balance calls height() twice
  - Computing height requires visiting every node
    - (linear in the size of the subtree)
- **Idea**: Store height of each node at the node
  - **Better idea**: Store balance factor (only requires 2 bits)

Enforcing the AVL Constraint

maintaining _parent makes it possible to traverse up the tree (helpful for rotations), but is not possible in an immutable tree.

```
class AVLNode[K, V](
  var _key: K,
  var _value: V,
  var _parent: Option[AVLNode[K,V]],
  var _left: AVLNode[K,V],
  var _right: AVLNode[K,V],
  var _isLeftHeavy: Boolean,  // true if balance(this) == -1
  var _isRightHeavy: Boolean, // true if balance(this) == 1
)
```

$$balance(n) = \begin{cases} -1 & \textbf{if } \texttt{n.\_isLeftHeavy} = \mathbf{T} \\ +1 & \textbf{if } \texttt{n.\_isRightHeavy} = \mathbf{T} \\ 0 & \textbf{otherwise} \end{cases}$$

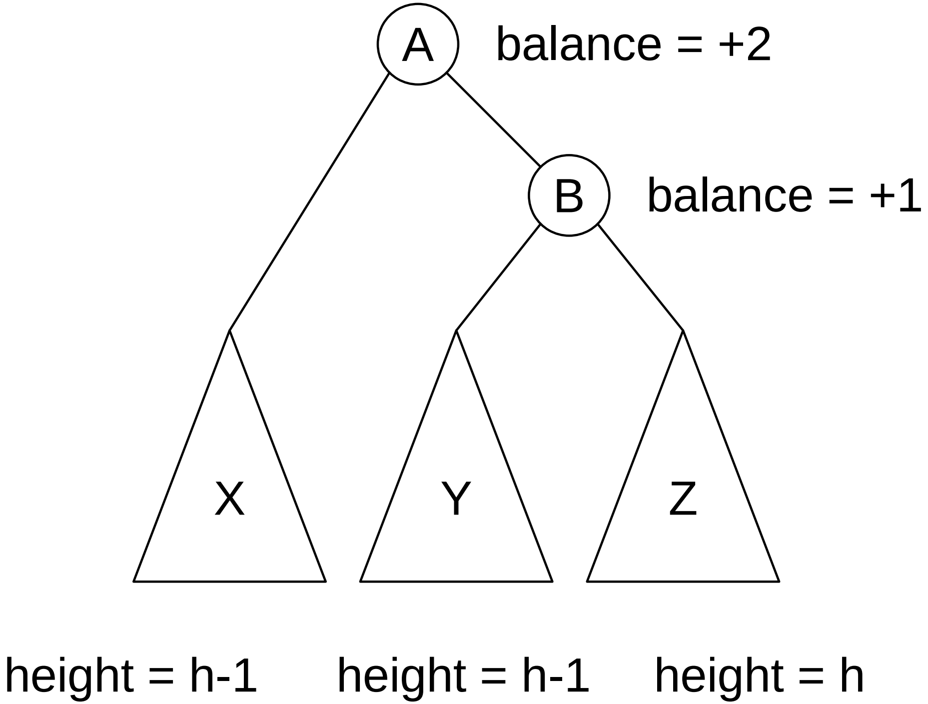
Enforcing the AVL Constraint

- Left Rotation
  - Before
    - **(A)** root; balance(**A**) = +2 (<u>too</u> right heavy)
    - **(B)** root.right; balance(**B**) = +1 (right heavy)

1) Left subtree of **(B)** becomes right subtree of **(A)**.
2) **(A)** becomes left subtree of **(B)**
3) **(B)** becomes root

  - After
    - balance(**A**) = 0, balance(**B**) = 0

Enforcing the AVL Constraint



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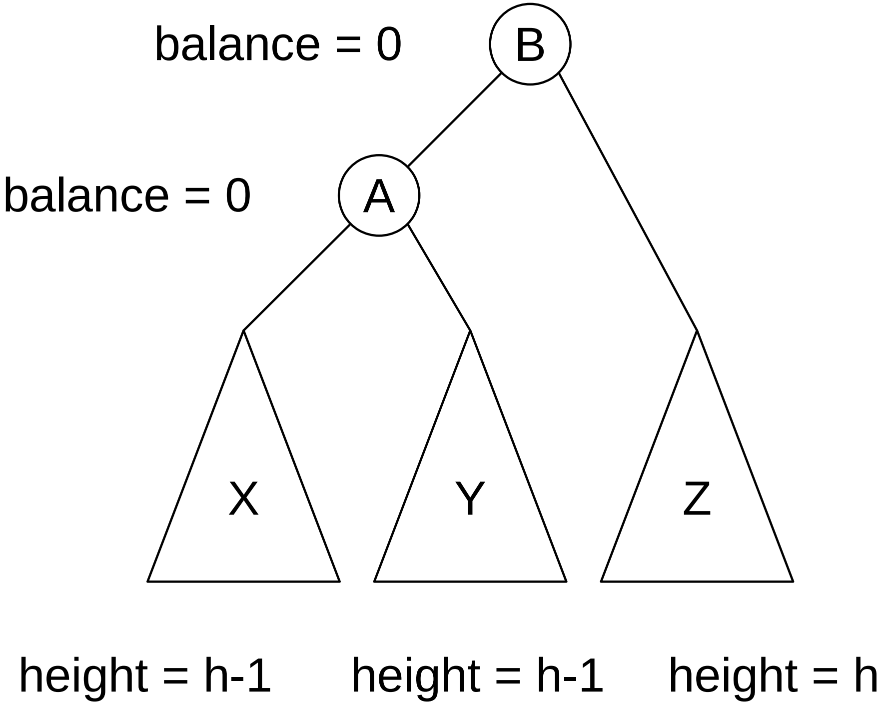


Enforcing the AVL Constraint

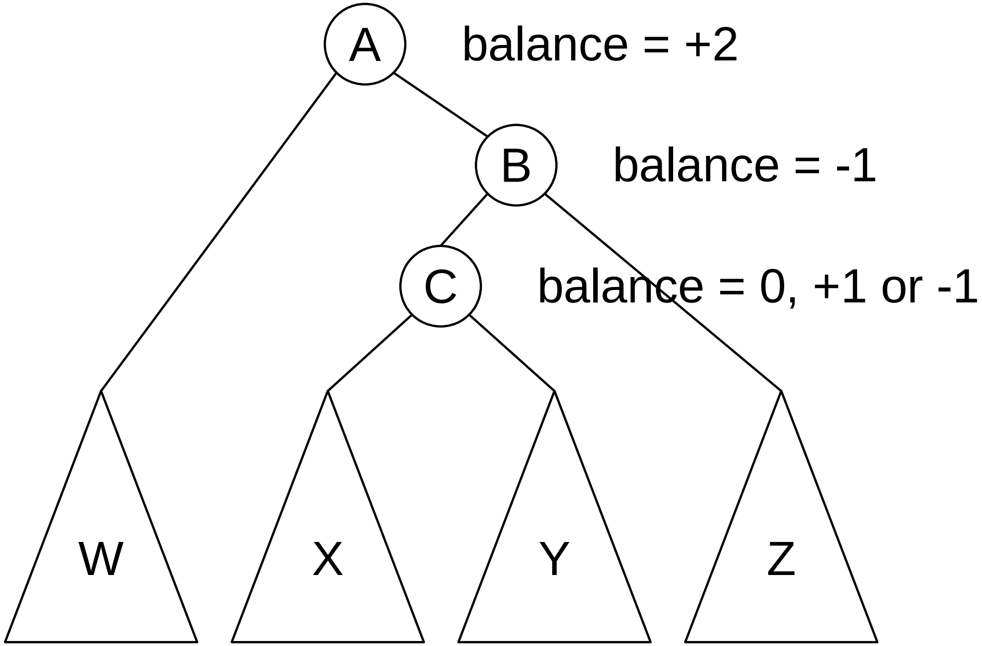
- Right-Left Rotation
  - Before
    - **(A)** root; balance(**A**) = +2 (too right heavy)
    - **(B)** root.right; balance(**B**) = -1 (left heavy)
    - **(C)** right.left.right

1) Left subtree of **(C)** becomes right subtree of **(A)**.
2) Right subtree of **(C)** becomes left subtree of **(B)**.
3) **(A)** becomes left subtree of **(C)**
4) **(B)** becomes right subtree of **(C)**
5) **(C)** becomes root

Enforcing the AVL Constraint

- After
  - if **(C)**'s BF was originally 0
    - **(A)** BF = 0; **(B)** BF = 0; **(C)** BF = 0
  - if **(C)**'s BF was originally -1
    - **(A)** BF = 0; **(B)** BF = +1; **(C)** BF = 0
  - if **(C)**'s BF was originally +1
    - **(A)** BF = -1; **(B)** BF = 0; **(C)** BF = 0

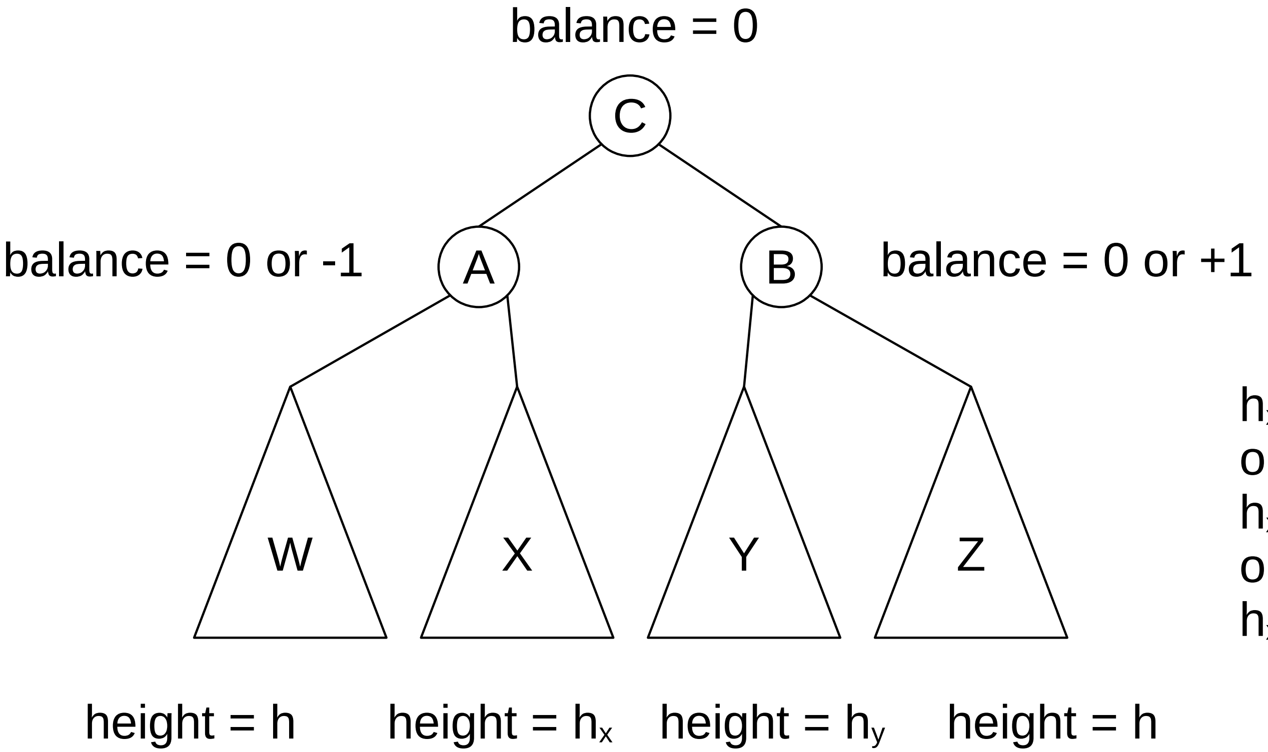
Enforcing the AVL Constraint



height = h height = h_x height = h_y height = h

$h_x = h_y = h$
or
$h_x = h - 1; h_y = h$
or
$h_x = h; h_y = h - 1$

Enforcing the AVL Constraint



$h_x = h_y = h$
or
$h_x = h - 1; h_y = h$
or
$h_x = h; h_y = h - 1$

Enforcing the AVL Constraint

- Rotate Right
  - Symmetric to rotate left
- Rotate Left-Right
  - Symmetric to rotate right-left

Inserting Records

- Inserting Records
  - Find insertion as in BST
  - Set balance factor of new leaf to 0
    - _isLeftHeavy = _isRightHeavy = false
  - Trace path up to root, updating balance factor
    - Rotate if balance factor off

Inserting Records

```
def insert[K, V](key: K, value: V, root: AVLNode[K, V]): Unit =
{
  var node = findInsertionPoint(key, root)
  node._key = key;   node._value = value
  node._isLeftHeavy = node._isRightHeavy = false
  while(node._parent.isDefined){
    if(node._parent._left == node){
      if(node._parent._isRightHeavy){
        node._parent._isRightHeavy = false; return
      } else if(node._parent._isLeftHeavy) {
        if(node._isLeftHeavy){ node._parent.rotateRight() }
        else { node._parent.rotateLeftRight() }
        return
      } else {
        node._parent.isLeftHeavy = true
      }
    } else {
      /* symmetric to above */
    }
    node = node._parent
} }
```

O(d) = O(log(n))

O(d) = O(log(n)) loops

O(1) per loop

Total Runtime = O(log(n))

Removing Records

- Removing Records
  - Remove the node
    - Find the node containing the value as in BST
      - If it doesn’t exist, return false
    - If the node is a leaf, remove it
    - If the node has one child, the child replaces the node
    - If the node has two children
      - copy smaller child value into node
      - remove smaller child node
  - Fix balance factors
    - Inverse of insertion

Maintaining Balance

- **Claim**: Only the balance factors of ancestors are impacted
  - The height of a node is only affected by its descendents
- **Claim**: Only one rotation will fix any remove/insert imbalance
  - Insert/remove change the height by at most one
- Only log(n) rotations are required for any insert/remove
  - Insert/remove are still log(n)